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$$\Omega \xrightarrow{\quad f \quad} \mathbb{K} \ \mid \|f\|_p := \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty.$$

Influential Mathematicians in Quantum: The Life and Work of Henri Léon Lebesgue.

About US

Quantum Formalism (QF) is a not-for-profit spin-out from Zaiku Group, dedicated to providing accessible online courses that introduce advanced mathematical topics to a diverse group of STEM professionals. Our aim is to support those looking to break into the nascent quantum computing industry or other emerging deep-tech fields, such as AI and privacy-enhancing technologies (PETs).

Our mission is to develop and deliver an online mathematics curriculum that empowers the next generation of algorithms and software innovators. We strive to offer mathematical training to exceptional individuals with the potential to become leading technical experts in their fields. Most of our courses are free, including our 'Abstract Mathematics 101 Bootcamp' series, which runs from May to December each year (learn more here).

In response to community requests, we have recetly started offering paid 'Specialisation' courses. These courses focus on specific topics like Machine Learning, Quantum Hardware and Cryptography and have a small nominal fee to help cover the costs associated with an enhanced learning experience.

Join our first specialisation bootcamp: "**Advanced Linear Algebra for Machine Learning**." This beginner-friendly yet rigorous course helps learners master essential mathematics for machine learning through hands-on practice. Download the course brochure from the landing page (link below) and sign up as soon as possible because we only have limited places open.

- **Advanced Linear Algebra for Machine Learning**: https://quantumformalism.com/advanced-linear-algebra-for-ml. Get a **70% discount** if you fill our survey 'Mathematical Skills Gap Among ML Practitioners'.

You can read more about the QF's journey via our recent Substack post 'Quantum Formalism: The Story Behind'.

Henri Léon Lebesgue

(28 June 1875 - 26 July 1941)

1. Early Childhood Summary

Henri Léon Lebesgue was born on 28 June 1875 in Beauvais. His father, who came from a humble background, managed to work his way up to become a typographer. Sadly, he passed away from tuberculosis shortly after Henri's birth, along with Henri's two older sisters. Throughout his life, Henri would continue to suffer from the long-term effects of the illness, leaving him with fragile health.

Lebesgue's mother was a tireless worker who was determined to ensure her son continued his studies, keeping him under her care for several more years. Lebesgue, who excelled academically from a young age, progressed through his education on a series of scholarships—from primary school to lycée, then to the preparatory classes at the prestigious Lycée Louis-Le-Grand.

During Lebesgue's early formative years, France was transitioning into the early stages of the Third Republic, established in 1870 following the Franco-Prussian War and the collapse of the Second Empire. This period was marked by political instability, with competing monarchist, republican, and Bonapartist factions. Over time, republican values such as secularism, democracy, and the principles of liberté, égalité, fraternité became entrenched, especially through educational reforms that aimed to secularise and democratise the education system. These developments would play a crucial role in shaping the intellectual and political landscape of the country.

Education reforms, notably those introduced by Jules Ferry in the 1880s, significantly expanded public access to education, making schooling free, compulsory, and secular. This reform allowed students like Lebesgue, from humble backgrounds, to advance through the academic system via scholarships. Lebesgue's eventual attendance at elite institutions such as the Lycée Louis-Le-Grand and later to the École Normale Supérieure placed him among the brightest minds of France. The intellectual environment during this time, especially in the fields of mathematics and science, was thriving, with giant figures such as Henri Poincaré leading important advancements in mathematics.

2. Early Academic Journey

By the time Lebesgue entered the prestigious École Normale Supérieure in Paris in 1894, France was experiencing profound industrial and social transformations. Urbanisation was accelerating, with cities such as Paris emerging as key centres of industry and culture. However, alongside economic growth and industrialisation, deep social divisions persisted, particularly between the affluent bourgeoisie and the working class. While social mobility remained constrained, the educational reforms of the era provided opportunities for talented individuals from modest backgrounds, such as Lebesgue, to ascend into academic and professional spheres.

Culturally, the Belle Époque (1871–1914) marked a period of optimism, innovation, and artistic flourishing, with Paris at the forefront of global culture. This era saw remarkable advancements in science and intellectual thought, particularly in mathematics and physics, fostering an environment that nurtured thinkers such as Lebesgue. Yet, despite these cultural and scientific strides, public health challenges, including widespread tuberculosis, continued to afflict many. Lebesgue’s fragile health throughout his life served as a poignant reminder of the era's limitations in medical care, even as science progressed rapidly in other fields.

Three years after entering the École Normale Supérieure, Lebesgue was awarded his teaching diploma in mathematics in 1897. For the following two years, he immersed himself in its library, where he studied Baire's papers on discontinuous functions. It was during this time that Lebesgue recognised the potential for significant advances in this area. In 1899, Lebesgue was appointed as a professor at the Lycée Centrale in Nancy, where he taught until 1902.

During this period, Lebesgue built upon the work of mathematicians such as Émile Borel and Camille Jordan. In 1901, he developed the theory of measure, and in his landmark paper Sur une généralisation de l'intégrale définie, published in the Comptes Rendus on 29 April 1901, he introduced the Lebesgue integral. This definition extended the concept of the Riemann integral by allowing for the integration of a broader class of functions, including many that are discontinuous. This generalisation marked a major breakthrough in the field, revolutionising integral calculus.

3. The Founder of Measure Theory

Before Lebesgue's contributions, the field of mathematical analysis was largely restricted to continuous functions, with integration methods heavily dependent on the Riemann integral. The Riemann approach worked well for simple, well-behaved functions, but it struggled with more complex, discontinuous functions and sets. This limitation left significant gaps in the ability to analyse irregular or fragmented functions, constraining the scope of mathematical analysis at the time.

Lebesgue's ground-breaking work revolutionised the field by introducing a new way of measuring sets and integrating functions, known today as the Lebesgue measure and Lebesgue integral respectively. Rather than relying on partitioning the domain into fixed intervals as in the Riemann method, Lebesgue focused on measuring the "size" of the set where the function takes specific values. This shift in perspective allowed for the integration of a much broader class of functions, including those with discontinuities and other complexities that were previously intractable.

Lebesgue's contribution stands as one of the great achievements of modern analysis, significantly broadening the scope of Fourier analysis. His ground-breaking work was encapsulated in his seminal doctoral dissertation entitled '**Intégrale, Longueur, aire**', which he presented to the Faculty of Science in Paris in 1902. The 130-page dissertation was published in the Annali di Matematica Pura in Milan that same year. Upon completing his doctorate, Lebesgue secured his first university position, becoming maître de conférences in mathematics at the Faculty of Science in Rennes in 1902. This appointment followed the traditional French academic path, whereby young scholars initially took up posts in the provinces before later receiving recognition with junior appointments in Paris.

Lebesgue's development of measure theory and the Lebesgue integral was a crucial turning point that directly influenced the early development of functional analysis and later the rigorous mathematical foundations of probability theory. By providing a framework for integrating functions over complex and irregular sets, Lebesgue's work allowed for a more precise understanding of function spaces, which became the cornerstone of functional analysis. In functional analysis, spaces of functions—such as **L^p** spaces, defined using Lebesgue integrals—became essential for studying operators and their properties. This foundational work in measure theory also paved the way for Andrey Kolmogorov's formalisation of probability theory in the 20th century. Kolmogorov used measure theory to define probability spaces, where probabilities were viewed as measures on sets of outcomes, making probabilistic concepts rigorous and mathematically sound.

4. The Move to Sorbonne

In 1910, Lebesgue was appointed maître de conférences in mathematical analysis at the Sorbonne. During the First World War, he contributed to France's defence efforts, though this period saw him fall out with Borel, who was engaged in similar work. Lebesgue continued in his role at the Sorbonne until 1918, when he was promoted to Professor of the Application of Geometry to Analysis. His academic career reached new heights in 1921 when he was appointed Professor of Mathematics at the Collège de France, a prestigious position he held until his death in 1941.

In addition to his work at the Collège de France, Lebesgue taught at the École Supérieure de Physique et de Chimie Industrielles de la Ville de Paris from 1927 to 1937, as well as at the École Normale Supérieure in Sèvres. His influence extended across multiple institutions, reflecting his lasting impact on French mathematics and his dedication to both teaching and research throughout his career.

Lebesgue made significant contributions to various other branches of mathematics, including topology, potential theory, the Dirichlet problem, the calculus of variations, set theory, surface area theory, and dimension theory. By 1922, when he published '**Notice sur les travaux scientifique de M. Henri Lebesgue**', he had authored nearly 90 books and papers. This 92-page work offers a detailed analysis of the key themes and results in his papers, reflecting the breadth and depth of his contributions to modern mathematics.

After 1922, while Lebesgue continued to be active in the mathematical community, his focus shifted towards educational and pedagogical concerns, as well as historical research and elementary geometry. Though his later works may not have had the ground-breaking impact of his earlier achievements, they demonstrate his enduring commitment to both the teaching of mathematics and the preservation of its historical foundations.

From his lecture courses, Lebesgue produced two influential monographs: '**Leçons sur l'intégration et la recherche des fonctions primitives**' (1904) and '**Leçons sur les séries trigonométriques**' (1906). These works played a crucial role in spreading his revolutionary ideas more widely. However, his contributions were not universally welcomed; many classical analysts, particularly in France, met his work with resistance and scepticism. Nevertheless, Lebesgue's career continued to progress, and in 1906, he was appointed to the Faculty of Science in Poitiers.

One of the early honours Lebesgue received was the prestigious invitation to deliver the Cours Peccot at the Collège de France, which he did in 1903. He was invited once again to present the same course two years later, in 1905.

6. Mathematical Influence in Quantum

Henri Lebesgue's contributions to functional analysis, particularly through his development of the Lebesgue integral, played a significant though indirect role in shaping the mathematical framework of quantum mechanics. His generalisation of integration extended beyond the limitations of the Riemann integral, allowing the analysis of a broader class of functions, including those that are discontinuous or partially continuous. This innovation was critical for modern analysis and laid the foundation for function spaces such as the $\mathbf{L^2}$ space, where square-integrable functions—functions whose square is integrable in the Lebesgue sense—reside.

In the 1920s, when Erwin Schrödinger formulated his wave mechanics, he unknowingly relied on the structure of these square-integrable functions to describe quantum states, forming what is now known as a Hilbert space, a cornerstone in quantum theory. Hilbert spaces provide the natural setting for the mathematical formulation of quantum mechanics, where physical observables are represented as operators acting on these spaces. Lebesgue's integral was essential for handling the probabilistic interpretation of the wavefunction, as the square of the wavefunction's modulus represents a probability density that must be integrable in the Lebesgue sense. While Schrödinger did not explicitly acknowledge Lebesgue's work, his reliance on these functional spaces is closely tied to the advancements in integration theory made by Lebesgue, which ultimately underpinned the rigorous mathematical treatment of quantum mechanics.

This case highlights a recurring theme in the history of physics, where physicists, driven by the need to describe natural phenomena, often developed theories and performed calculations without fully appreciating the underlying mathematical structure. In Schrödinger's case, the use of $\mathbf{L^2}$ (square-integrable) spaces, defined using the Lebesgue integral, allowed him to describe quantum states accurately, but the deeper intricacies of measure theory, which underpinned the mathematics of these spaces, were largely unknown to him. Later, John von Neumann formalised the mathematical framework of quantum mechanics in his work 'Mathematical Foundations of Quantum Mechanics' (1932), rigorously defining quantum states as vectors in Hilbert space and quantum observables as operators acting on those spaces. Von Neumann's formulation not only recognised the crucial role of functional analysis and measure theory but also provided a more complete and mathematically unified interpretation of quantum mechanics.

This disconnect between the practical use of mathematics and its formal foundations is a testament to the way physicists' naïve intuition can lead to significant discoveries, even when the theoretical underpinnings are not fully understood at the time. Von Neumann's work effectively bridged this gap, offering a rigorous mathematical structure to Schrödinger's wave mechanics approach to quantum mechanics and Heisenberg's alternative matrix mechanics approach.

References

- **The main source used to create this brief:** https://mathshistory.st-andrews.ac.uk/Biographies/Lebesgue/.

Reading recommendation

- 'Lebesgue's Theory of Integration: Its Origins and Development'.

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